JOURNAL OF APPROXIMATION THEORY 36, 226–236 (1982)

Representers of Linear Functionals, Norm-Attaining Functionals, and Best Approximation by Cones and Linear Varieties in Inner Product Spaces

FRANK DEUTSCH*

Department of Mathematics, The Pennsylvania State University, University Park, Pennsylvania 16802, U.S.A.

Communicated by Richard S. Varga

Received August 6, 1981

In an inner product space X, a cone or a linear variety which is generated by a finite number of linear functionals in the dual space X^* is a Chebyshev set (i) if and only if each of the generating functionals attains its norm, and (ii) if and if each of them has a "Riesz representer" in X.

1. INTRODUCTION

A fundamental property of Hilbert space (i.e., a complete inner product space) is that *every* closed (nonempty) convex subset is a "Chebyshev" set (i.e., each point of the whole space has a unique nearest point in the set). Using this property, one can give simple proofs of the important projection theorem and the Fréchet–Riesz representation theorem, among others.

However, in an inner product space which is *not* complete, this property no longer holds. Indeed, in such a space, there always exist closed linear subspaces (even hyperplanes) which are not Chebyshev. This raises the question: "How does one recognize which closed convex subsets of an inner product space are Chebyshev?"

One reason this seems to be an important question is the following. In many applications of best approximation by functions which arise in the engineering sciences, the natural setting is a space of real-valued *continuous* functions on some interval $[a, b]$. If such a space is endowed with the inner product $\langle x, y\rangle = \int_a^b x(t)\, y(t)\, dt$, this space (denote it $C_2[a, b]$) is *not* complete. (Its completion can be identified with the space $L_2[a, b]$ of all

* Supported in part by a grant from the National Science Foundation.

0021-9045/82/110226-11$02.00/0

square-integrable functions on $[a, b]$.) Suppose one is seeking best approximations to a given continuous function x from the subspace M of all continuous functions having zero mean and zero first moment:

$$M = \left\{ y \in C_2[a, b] \;\middle|\; \int_a^b y(t)\,dt = 0 = \int_a^b ty(t)\,dt \right\}.$$

Is M a Chebyshev subspace? The Hilbert space theory cannot be directly applied in this example. While M is a closed subspace in $C_2[a, b]$, it is *not* closed in the larger space $L_2[a, b]$, and hence certainly not Chebyshev in $L_2[a, b]$. One possible way around this particular difficulty could be to enlarge M by replacing it by its closure $\tilde{M}$ in $L_2[a, b]$. However, $\tilde{M}$ contains many discontinuous functions and although $\tilde{M}$ is a Chebyshev set in $L_2[a, b]$, we have no a priori guarantee that the best approximation in $\tilde{M}$ to a given (continuous) $x \in C_2[a, b]$ will be *continuous*, i.e., in M.

We summarize briefly. Suppose we are given an approximation problem in an incomplete inner product space. It is not generally a satisfactory procedure to embed the problem in the Hilbert space completion of the space in question. What we really would like is a useful condition which allows us to conclude exactly when a given closed convex subset of an (incomplete) inner product space is Chebyshev.

It is well-known that each finite-dimensional subspace, or any closed convex subset thereof, in an inner product space is Chebyshev. The main result of this article is a simple and useful characterization of Chebyshevness for convex sets in a certain class which includes the subspaces of finite codimension. For example, Corollary 3.2 and Theorem 3.3 below can be combined to yield the following result.

Let X be an inner product space, $\{x_1^*, x_2^*,..., x_n^*\}$ a linearly independent set of functionals in the dual space X^*, $\alpha_1, \alpha_2,..., \alpha_n$, n real numbers, and let C denote either one of the following two sets:

$$C = \bigcap_1^n \{x \in X \mid x_i^*(x) \geqslant \alpha_i\} \qquad \text{(generalized cone)}$$

or

$$C = \bigcap_1^n \{x \in X \mid x_i^*(x) = \alpha_i\} \qquad \text{(finite codimensional variety).}$$

Then the following statements are equivalent: (1) *C is Chebyshev*; (2) *C is proximinal*; (3) *Each x_i^* attains its norm*; (4) *Each x_i^* has a representer in X.*

(Precise definitions will be given below.) Incidentally, as a consequence of this result, the subspace M of $C_2[c, b]$ discussed above is immediately seen to be Chebyshev because the two functionals $x_1^*(y) := \int_a^b y(t)\,dt$ and $x_2^*(y) := \int_a^b ty(t)\,dt$ have representers 1 and t in $C_2[a, b]$.

2. Norm-Attaining Functionals and Representers of Linear Functionals

In this section we give the relevant definitions and establish a key link (Lemma 2.2) in the chain holding the main results of Section 3 together.

Throughout the remainder of the paper, X *will always denote an inner product space* with inner product $\langle \cdot, \cdot \rangle$, norm $\|x\| = \sqrt{\langle x, x \rangle}$, and dual space X^*: the space of all bounded linear functionals on X. We assume the scalar field is real, although the results are valid in the complex case as well. (Some obvious, but minor, modifications need to be made when dealing with inequalities.)

Let K be a nonempty convex subset of X and $x \in X$. An element $y_0 \in K$ is a *best approximation* to x provided

$$\|x - y_0\| = d(x, K) := \inf_{y \in K} \|x - y\|.$$

K is called *proximinal* (resp. *Chebyshev*) provided each element of X has at least (resp. exactly) one best approximation in K. Because X is strictly convex, each $x \in X$ has at most one best approximation in K. We denote it by $P_K x$. In particular, *K is proximinal if and only if K is Chebyshev.*

If M is a linear subspace, the *orthogonal complement* of M is the set

$$M^\perp := \{x \in X \mid \langle y, x \rangle = 0 \text{ for all } y \in M\}.$$

Given any $x \in X$, define a functional x^* on X by

$$x^*(y) = \langle y, x \rangle \qquad \text{for all} \quad y \in X. \tag{2.1}$$

It is an elementary fact that x^* is linear, bounded, and

$$\|x^*\| = \|x\|. \tag{2.2}$$

Thus every $x \in X$ gives rise to a functional $x^* \in X^*$ in this natural way. Conversely, if $x^* \in X^*$ has representation (2.1), we call x a *representer* of x^*. Clearly, a functional in X^* can have at most one representer in X. If X is complete, then the classical Fréchet–Riesz representation theorem (see, e.g., [4, p. 249]) states that *every* $x^* \in X^*$ has a representer in X. However, if X is not complete, then there always exist functionals in X^* which do not have representers in X. (In fact, if $\{x_n\}$ is a Cauchy sequence in X which does not converge, then $x^*(x) := \lim\langle x, x_n\rangle$ defines a functional $x^* \in X^*$ which has no representer in X.)

A nonzero functional $x^* \in X^*$ is said to *attain its norm* if there exists $z \in X$ with $\|z\| = 1$ and $x^*(z) = \|x^*\|$. By the strict convexity of X, if x^* attains its norm, it does so at a unique point z.

We record first the following well-known facts about hyperplanes for future reference.

2.1 THEOREM. *Let $0 \neq x^* \in X^*$ and $M = \ker x^* := \{x \in X \mid x^*(x) = 0\}$.*

(1) *Then $d(x, M) = (1/\|x^*\|)\, |x^*(x)|$ for all $x \in X$.*

(2) *The following statements are equivalent.*

(a) *M is Chebyshev;*

(b) *M is proximinal;*

(c) *Some $x \in X \backslash M$ has a best approximation in M;*

(d) *x^* attains its norm.*

Moreover, if x^* attains its norm at z, then $z \in M^\perp$ and $P_M(x) = x - (x^*(x)/\|x^*\|)\, z$ for every $x \in X$.

Remark. Actually, (1) and the equivalence of (b), (c), and (d) of (2) are valid in *any* normed linear space X (see, e.g., [6]).

The next result adds yet another equivalent condition to Theorem 2.1(2).

2.2 LEMMA. *Let $x^* \in X^* \backslash \{0\}$. Then x^* attains its norm if and only if x^* has a representer in X.*

Proof. Suppose x^* attains its norm at z: $\|z\| = 1$ and $x^*(z) = \|z\|$. Set $M = \ker x^*$. Then by 2.1(2), M is Chebyshev, $x = P_M(x) + (x^*(x)/\|x^*\|)\, z$ for each $x \in X$, and $z \in M^\perp$. Hence

$$\begin{aligned}\langle x, \|x^*\|\, z\rangle &= \|x^*\| \left\langle P_M x + \frac{x^*(x)}{\|x^*\|} z, z \right\rangle \\ &= x^*(x) \qquad \text{for each} \quad x \in X.\end{aligned}$$

Thus $\|x^*\|\, z$ is a representer for x^*.

Conversely, let $x \in X$ be a representer of x^* and set $z = x/\|x\|$. Then $\|z\| = 1$ and

$$x^*(z) = \left\langle \frac{x}{\|x\|}, x \right\rangle = \|x\| = \|x^*\|.$$

Thus x^* attains its norm at z. ∎

Using the fact that X is dense in its Hilbert space completion and that X may be identified with a subspace of X^* via the mapping $x \mapsto \langle \cdot, x \rangle$, we immediately obtain from Lemma 2.2

2.3 COROLLARY. *The set of all norm-attaining functionals in X^* is a dense linear subspace.*

It is of some interest to compare this corollary with the powerful Bishop–Phelps theorem [2] asserting—for any complete normed linear space X—the denseness of the set of all norm-attaining functionals in X^*. In this general situation however, the dense subset of norm-attaining functionals need *not* form a linear subspace.

Recall that a (convex) *cone* is a convex set C with the property that $\lambda x \in C$ whever $x \in C$ and $\lambda \geqslant 0$. The *conical hull* of a set S, denoted $\text{con}(S)$, is the set of all non-negative linear combinations of elements of S:

$$\text{con}(S) = \left\{ \sum_i \lambda_i x_i \mid \lambda_i \geqslant 0, x_i \in S \right\}.$$

The conical hull of any set is clearly a cone. A cone C is called *finitely generated* if it is the conical hull of a finite set. The *polar* of a cone C is the cone

$$C^0 := \{x \in X \mid \langle y, x \rangle \leqslant 0 \text{ for each } y \in C\}.$$

Observe that if M is a linear subspace, then $M^0 = M^\perp$.

2.4 LEMMA. (1) *If C is a finitely generated cone, then C is Chebyshev.*

(2) *If C is a Chebyshev cone, then C^0 is also a Chebyshev cone.*

Proof (Sketch). (1) C is a convex subset of a finite-dimensional subspace so it suffices to prove that C is closed. If the generating elements for C form a linearly independent set, then the proof that C is closed follows just like the proof that finite-dimensional subspaces are closed. If the generating set S is linearly dependent, then each positive linear combination of elements of S can be reduced to a positive linear combination of a linearly *independent* subset of S. This essentially reduces the problem to the case when S is linearly independent.

(2) In [1, Theorem 1, p. 18], Aubin showed that if C is a closed cone in a *Hilbert* space, then (C and C^0 *are Chebyshev* since they are closed and convex and)

$$x = P_C x + P_{C^0} x \qquad \text{for every} \quad x \in X. \tag{2.4.1}$$

However, a close inspection of Aubin's proof reveals that completeness of X is not necessary; all that is really essential is that C be Chebyshev and the well-known characterization of best approximations from convex cones (see, e.g., [1, Lemma 1, p. 18]). In this way, we can conclude that for every $x \in X$, $x - P_C x$ is a best approximation to x from C^0. Hence (2.4.1) holds and, in particular, C^0 is Chebyshev. ∎

3. Characterizing Cones and Varieties Which Are Chebyshev

Our first result is valid for certain cones.

3.1 Theorem. *Let $\{x_i^* \mid i = 1, 2,..., n\}$ be a finite subset of X^* such that for some $x_0 \in X$, $x_i^*(x_0) > 0$ for all i. Let C denote the cone*

$$C = \bigcap_1^n \{x \in X \mid x_i^*(x) \geqslant 0\}.$$

Then the following statements are equivalent.

(1) *C is Chebyshev;*

(2) *C is proximinal;*

(3) *Each x_i^* attains its norm;*

(4) *Each x_i^* has a representer in X.*

Proof. The equivalence of (1) and (2) was already noted in Section 2 and the equivalence of (3) and (4) follows from Lemma 2.2.

(4) ⇒ (1). If each x_i^* has a representer $x_i \in X$, then

$$C = \bigcap_1^n \{x \mid \langle x, x_i \rangle \geqslant 0\}.$$

Define

$$K = \text{con}\{-x_i \mid i = 1, 2,..., n\}.$$

Then K is a finitely generated cone, and it is easy to verify that $K^0 = C$. By Lemma 2.4(1), K is Chebyshev and by 2.4(2), $K^0 = C$ is Chebyshev.

(1) ⇒ (4). Suppose C is Chebyshev. We must show that each x_i^* has a representer in X. We may assume $\|x_i^*\| = 1$ for each i. First note that if any x_i^* is in the conical hull of the others, $x_i^* = \sum_{j \neq i} \lambda_j x_j^*$, for some $\lambda_j \geqslant 0$, then x_i^* may be eliminated from the definition of C without changing C. In addition, if each $x_j^*, j \neq i$, has a representer $x_j \in X$, then $x_i^* = \sum_{j \neq i} \lambda_j x_j^*$ has

the representer $\sum_{j\neq i}\lambda_j x_j$. These remarks show that it is no loss of generality to assume that no x_i^* is in the conical hull of the others:

$$x_i^* \notin C_i := \text{con}\{x_j^* \mid j \neq i\} \qquad (i = 1, 2,..., n).$$

Fix any index $i \in \{1, 2,..., n\}$. Since $x_i^* \notin C_i$ and C_i is weak* closed, the separation theorem [4, p. 417] implies that there exists $x_i \in X$ such that

$$x_i^*(x_i) < \inf\{x^*(x_i) \mid x^* \in C_i\}.$$

Since C_i is a cone, the infimum on the right must be 0. In particular,

$$x_i^*(x_i) < 0 \leqslant x_j^*(x_i) \qquad \text{for all} \quad j \neq i.$$

Let $y_i = x_i - (x_i^*(x_i)/x_i^*(x_0))\, x_0$. Then

$$x_i^*(y_i) = 0 < x_j^*(y_i) \qquad \text{for all} \quad j \neq i$$

and thus $y_i \in C$. Choose any $0 < \varepsilon < 1$ with $2\varepsilon < x_j^*(y_i)$ for all $j \neq i$.

Claim 1. $B_\varepsilon(y_i) \cap \ker x_i^* \subset C$.

(Here, and for the rest of the proof, $B_\varepsilon(z)$ will denote the open ball centered at z with radius ε.) For let $y \in B_\varepsilon(y_i) \cap \ker x_i^*$. Then for any $j \neq i$,

$$x_j^*(y) = x_j^*(y - y_i) + x_j^*(y_i) > -\varepsilon + 2\varepsilon > 0$$

so $y \in C$.

Next choose any $x \in B_{\varepsilon/4}(y_i)$ such that $x_i^*(x) < 0$. Then for all $j \neq i$,

$$x_j^*(x) = x_j^*(y_i) + x_j^*(x - y_i) > 2\varepsilon - \varepsilon/4 > 0.$$

Claim 2. $P_C x \in \ker x_i^*$.

For if not, then $x_i^*(P_C x) > 0$. For each $\lambda \in [0, 1]$ define $z_\lambda = \lambda x + (1-\lambda) P_C x$. Then $x_i^*(z_0) > 0 > x_i^*(z_1)$ so there exists $0 < \lambda_0 < 1$ such that $x_i^*(z_{\lambda_0}) = 0$. Then $z_{\lambda_0} \in C$ and

$$\|x - z_{\lambda_0}\| = (1-\lambda_0)\,\|x - P_C x\| < \|x - P_C x\|$$

which is impossible.

Claim 3. $P_C x \in B_{\varepsilon/2}(y_i)$.

For $\|x - y_i\| < \varepsilon/4$ and $y_i \in C$ implies that $\|x - P_C x\| < \varepsilon/4$. Hence $\|P_C x - y_i\| < \varepsilon/2$.

By Claim 3 we can choose $0 < \varepsilon' < 1$ such that

$$B_{\varepsilon'}(P_C x) \subset B_{\varepsilon/2}(y_i).$$

Set $x' = \varepsilon' x + (1-\varepsilon') P_C x$. Then $P_C x' = P_C x$ and $x' \in B_{\varepsilon'}(P_C x')$. Also $x_i^*(x') < 0$ and $x_j^*(x') > 0$ for all $j \neq i$.

Claim 4. $d(x', C) \leqslant d(x', \ker x_i^*)$.

For let $y \in \ker x_i^*$. If $y \in B_\varepsilon(y_i)$, then by Claim 1, $y \in C$ so $\|x' - P_C x'\| \leqslant \|x' - y\|$. If $y \notin B_\varepsilon(y_i)$, then since

$$\|y_i - x'\| \leqslant \|y_i - x\| + \|x - x'\| < \varepsilon/4 + (1 - \varepsilon')\|x - P_C x\| < \varepsilon/2,$$

if follows that

$$\|x' - y\| \geqslant \|y - y_i\| - \|y_i - x'\| \geqslant \varepsilon - \varepsilon/2 > \|x - P_C x\| \geqslant \|x' - P_C x'\|.$$

Thus $\|x' - P_C x'\| \leqslant \|x' - y\|$ for all $y \in \ker x_i^*$ and the claim is proved.

Notice that $\ker x_i^*$ is a hyperplane which separates x' and C (since $x_i^*(x') < 0 \leqslant x_i^*(y)$ for all $y \in C$). Thus (using 2.1(1)) $d(x', \ker x_i^*) \leqslant d(x', C)$. It follows from Claim 4 that $d(x; \ker x_i^*) = d(x', C)$. By Claim 2, $P_C x' = P_C x$ is in $\ker x_i^*$ so $P_C x' = P_{\ker x_i^*} x'$. Since $x' \notin \ker x_i^*$, x_i^* attains its norm by Theorem 2.1(2) ((c) $\Leftrightarrow$ (d)). By Lemma 2.2, x_i^* has a representer in X. Since i was arbitrary, (4) follows. ∎

3.2 Corollary. *Let $\{x_i^* \mid i = 1, 2,..., n\}$ be a linearly independent subset of X^*, let $\alpha_1,..., \alpha_n$ be any n real numbers and let*

$$C = \bigcap_1^n \{x \in X \mid x_i^*(x) \geqslant \alpha_i\}.$$

Then the following statements are equivalent.

(1) *C is Chebyshev*;

(2) *C is proximinal*;

(3) *Each x_i^* attains its norm*;

(4) *Each x_i^* has a representer in X.*

Proof. Since the x_i^* are linearly independent, for any set of n scalars β_i, there exists $x \in X$ such that $x_i^*(x) = \beta_i$ for all i. In particular, there exist x_0, x_1 in X such that

$$x_i^*(x_0) = 1, \qquad x_i^*(x_1) = \alpha_i \qquad (i = 1, 2,..., n).$$

Let $C_0 = \bigcap_1^n \{x \in X \mid x_i^*(x) \geqslant 0\}$. Note that $C = C_0 + x_1$ and thus C is Chebyshev $\Leftrightarrow$ C_0 is Chebyshev. The result now follows from Theorem 3.1. ∎

An analogous result also holds for finite codimensional linear varieties and, in particular, for subspaces of finite codimension. This is the content of the next theorem.

3.3 THEOREM. *Let $\{x_1^*, x_2^*, ..., x_n^*\}$ be a linearly independent set in X^*, $\alpha_1, \alpha_2, ..., \alpha_n$ real numbers, and let V denote the linear variety*

$$V = \bigcap_1^n \{x \in X \mid x_i^*(x) = \alpha_i\}.$$

Then the following statements are equivalent.

(1) *V is Chebyshev;*

(2) *V is proximinal;*

(3) *Each x_i^* attains its norm;*

(4) *Each x_i^* has a representer in X.*

Proof. The equivalences $(1) \Leftrightarrow (2)$ and $(3) \Leftrightarrow (4)$ follow as in Theorem 3.1. Since the x_i^* are independent, $V \neq \varnothing$. Let $x_0 \in V$ and note that $V = V_0 + x_0$, where

$$V_0 = \bigcap_1^n \{x \in X \mid x_i^*(x) = 0\}.$$

Since V is Chebyshev $\Leftrightarrow$ V_0 is Chebyshev, we may assume that all $\alpha_i = 0$.

$(2) \Rightarrow (3)$. Suppose V is proximinal. By a result of Singer [7, Theorem 2.4, p. 12], the set

$$A := \{(x_1^*(x), x_2^*(x), ..., x_n^*(x)) \mid x \in X, \|x\| \leqslant 1\}$$

is closed in $\boldsymbol{R}^n$. Fix any index $i \in \{1, 2, ..., n\}$. Choose a sequence $\{x_k\}$ in X, $\|x_k\| \leqslant 1$, such that $x_i^*(x_k) \to \|x_i^*\|$ as $k \to \infty$. By passing to a subsequence, we may also assume that $\{x_j^*(x_k)\}$ converges for all $j \neq i$. Since A is closed, there exists $x_0 \in X$ such that $\|x_0\| \leqslant 1$ and $x_i^*(x_0) = \|x_i^*\|$. That is, x_i^* attains its norm. Since i was arbitrary, (3) follows.

$(4) \Rightarrow (1)$. Suppose each x_i^* has a representer $x_i \in X$. Then

$$V = \bigcap_1^n \{x \in X \mid \langle x, x_i \rangle = 0\}.$$

Let $N = \operatorname{span}\{x_1, x_2, ..., x_n\}$. It is easy to see that $N^\perp = V$. Since N is finite-dimensional, it is Chebyshev. By Lemma 2.4(2), $V = N^\perp = N^0$ is also Chebyshev. Thus (1) holds. ∎

Remark. As we noted in Theorem 2.1(2), in the special case when $n = 1$, the equivalence of (1), (2), and (3) is well-known.

4. A Partial Generalization

During the course of proving the implication (1) ⇒ (4) of Theorem 3.1 and (2) ⇒ (3) of Theorem 3.3, we have essentially proved the following partial generalizations. Let X be an arbitrary normed linear space, $\{x_1^*, x_2^*, \ldots, x_n^*\}$ a linearly independent subset of X^*, and let C denote the cone

$$C = \bigcap_1^n \{x \in X \mid x_i^*(x) \geqslant 0\}$$

and M the subspace of finite codimension

$$M = \bigcap_1^n \{x \in X \mid x_i^*(x) = 0\}.$$

Then a *necessary* condition that either C or M be proximinal is that each x_i^* attains its norm.

Thus it is natural to ask if this necessary condition is also always sufficient. We have shown this to be the case in any inner product space. Also, the condition is sufficient if X is any reflexive Banach space since in this case *every* closed convex subset is proximinal. Blatter and Cheney [3] and Pollul [5, Lemma 2.6] have essentially shown that in the space $X = c_0$, the condition is sufficient (for subspaces of finite codimension). However, the following example shows that the condition is *not sufficient* in the space $C[0, 1]$ of real-valued continuous functions on $[0, 1]$ with the supremum norm.

Example. Let

$$M = \{x \in C[0, 1] \mid x_1^*(x) = 0 = x_2^*(x)\},$$

where $x_1^*(x) := x(0)$ and $x_2^*(x) := \int_0^1 x(t)\,dt$. Then M is a closed subspace of codimension 2 in $C[0, 1]$ and both x_1^* and x_2^* attain their norm (at $z(t) \equiv 1$). Letting

$$X_0 = \{x \in C[0, 1] \mid x_1^*(x) = 0\},$$

we see that

$$M = \{x \in X_0 \mid x_2^*(x) = 0\}$$

is a closed hyperplane in X_0. If M is proximinal in $C[0, 1]$, then M is certainly proximinal in X_0 and hence x_2^* must attain its norm (*in* X_0!). That is, the restriction $x_0^* = x_2^*|_{X_0}$ must attain its norm. It is easy to see that

$\|x_0^*\| = 1$. But if $x_0 \in X_0$ and $\|x_0\| = 1$, then $x_0(0) = 0$ and, because of continuity,

$$x_0^*(x_0) = x_2^*(x_0) = \int_0^1 x_0(t)\,dt < 1.$$

Thus x_0^* fails to attain its norm which is a contradiction. This shows that M is not proximinal. (In fact, no element $x \in C[0, 1] \backslash M$ with $x(0) = 0$ has a best approximation in M.)

Finally, we should mention that various characterizations of proximinal subspaces, both in general and in special spaces, have been given by Holmes, Garkavi, Godini, Singer, and others (see, e.g., [7, pp. 12–13]).

REFERENCES

1. J.-P. AUBIN, "Applied Functional Analysis," Wiley–Interscience, New York, 1979.
2. E. BISHOP AND R. R. PHELPS, A proof that every Banach space is subreflexive, *Bull. Amer. Math. Soc.* **67** (1961), 97–98.
3. J. BLATTER AND E. W. CHENEY, On the existence of extremal projections, *J. Approx. Theory*, **6** (1972), 72–79.
4. N. DUNFORD AND J. T. SCHWARTZ, Linear Operators, I, Interscience, New York, 1958.
5. W. POLLUL, "Reflexivität und Existenz-Tielräume in der Linearen Approximationstheorie," Dissertation, Bonn, 1972.
6. I. SINGER, "Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces," Springer–Verlag, New York/Berlin, 1970.
7. I. SINGER, "The Theory of Best Approximation and Functional Analysis," CBMS Regional Conference Series No. 13, SIAM, Philadelphia, 1974.